Essential Spectrum of Fermionic Quantum Feild Model

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Abstract. In this paper, an abstract interacting system of fermionic quantum field is investigated. The state space is defined by a tensor product Hilbert space of a fermion Fock spce and an abstract Hilbert space. It is assumed that the total Hamiltonian is a self-adjoint operator on the state space. In the main theorem, the location of essential spectrum the total Hamiltonian is investigated. Its application to the Yukawa model, which is the system of Dirac field coupled to Klein-Gordon field, is also investigated.

1 Introduction and Main Results

For a mathematical analysis of quantum physics models, the state spaces are defined by a Hilbert space and Hamiltonians are given by linear operators on the Hilbert space. An interest of the analysis is to analyze the spectrum of the Hamiltonian. For the system of quantum mechanics, the spectrum properties have been analyzed, and in particular, the essential spectrum of Schrödinger operator is exactly analyzed by the HVZ theorem (See [3]). For a research on the essential spectrum of quantum filed model, the location of the essential spectrum for an abstract interacting system of bosonic field and its application non-relativistic QED model are investigated by Arai in [2]. In this paper, an abstract interacting system of fermionic quantum field is investigated. The state space is given by a tensor product Hilbert space of a fermion Fock space and an Hilbert space. The total Hamiltonian of the system consists of a free Hamiltonian and an interacting Hamiltonian. It is assumed that the Hamiltonian is a self-adjoint operator and bounded from below. By supposing some conditions of the weak commutator for interacting Hamiltonian and the creation-annihilation operators, the location of essential spectrum of the Hamiltonian are analyzed. The main result in this paper can be regarded as that of a fermionic version obtained by Arai in [2]. For an application of the main theorem to Yukawa model is also investigated. Yukawa model is the interacting system of Dirac fields coupled to Klein-Gordon fields. Its state space is defined by a boson-fermion Fock space. By applying the main theorem and the results obtained in [2] and [12], the essential spectrum of the Yukawa Hamiltonian is exactly identified and the HVZ theorem for Yukawa model is obtained. For research related to the essential spectrum of quantum filed model, the location of the essential spectrum of the concrete quantum field model are also exactly analyzed in [1] and [5]. For the analysis of Yukawa

model, it has been analyzed from the viewpoint of constructive quantum filed theory (refer to e.g. [8] and [9]). For other literature related to Yukawa model, see also [4], [6], [7].

Let us introduce an abstract interacting system of fermionic field. The state space is given by

$$\mathcal{H} \;=\; \mathcal{F}_{\mathrm{f}}(\mathcal{K}) \otimes \mathcal{T} \tag{1}$$

where $\mathcal{F}_{\mathrm{f}}(\mathcal{K})$ is a fermion Fock space on a Hilbert space $\mathcal{K}$ and $\mathcal{T}$ is a Hilbert pace. The definitions and basic properties of Fock spaces and their operators are explained in subsequent section. The free Hamiltonian is given by

$$H_0 \;=\; d\Gamma_{\mathrm{f}}(K) \otimes I \;+\; I \otimes T, \tag{2}$$

where $d\Gamma_{\mathrm{f}}(K)$ is a second quantization of the self-adjoint and non-negative operator K, and T is a self-adjoint operator on $\mathcal{T}$ with bounded from below. Then it is seen that H_0 is a self-adjoint on $\mathcal{D}(H_0) = \mathcal{D}(d\Gamma_{\mathrm{f}}(K) \otimes I) \cap \mathcal{D}(I \otimes T)$ with bounded from below. The total Hamiltonian is given by

$$H \;=\; H_0 \;+ H_{\mathrm{I}}, \tag{3}$$

where H_{I} is a symmetric operator on $\mathcal{H}$.

Here we review the the weak commutator which is introduced in [2]. Let X and Y be a densely defined linear operator on a Hilbert space $\mathcal{X}$. Assume that there exists a linear operator Z and a dense subspace $\mathcal{M}$ such that $\mathcal{M} \subset \mathcal{D}(Z) \cap \mathcal{D}(X) \cap \mathcal{D}(X^*) \cap \mathcal{D}(Y) \cap \mathcal{D}(Y^*)$ and for $\Phi, \Psi \in \mathcal{M}$,

$$(X^*\Phi, Y\Psi) - (Y^*\Phi, X\Psi) \;=\; (\Phi, Z\Psi).$$

Then the restriction Z to $\mathcal{M}$ is called a weak commutator of X and Y on $\mathcal{M}$ and it is denoted by $[X,Y]^0_{\mathcal{M}} \;=\; Z_{\lceil \mathcal{M}}$. As a remark, the weak commutator is denoted by $[X,Y]_{w,\mathcal{M}}$ in [2].

Here we assume the following conditions :

(A.1) H is self-adjoint on $\mathcal{D}(H) = \mathcal{D}(H_0) \cap \mathcal{D}(H_{\mathrm{I}})$ and bounded from below.

(A.2) For all $h \in \mathcal{D}(K)$, $[H_{\mathrm{I}}, B(h) \otimes I]^0_{\mathcal{D}(H)}$ and $[H_{\mathrm{I}}, B^*(h) \otimes I]^0_{\mathcal{D}(H)}$ exist where $B(h)$ and $B^*(h)$ denotes the annihilation operator and the creation operator, respectively. For any sequence $\{h_n\}_{n=1}^{\infty}$ of $\mathcal{D}(K)$ satisfying $\text{w-}\lim_{n\to\infty} h_n = 0$ and $\|h_n\| = 1$, $n \geq 1$, it follows that for all $\Psi \in \mathcal{D}(H)$,

$$\text{(1)} \quad \text{s-}\lim_{n\to\infty} [H_{\mathrm{I}}, B(h_n) \otimes I]^0_{\mathcal{D}(H)} \Psi = 0,$$

$$\text{(2)} \quad \text{s-}\lim_{n\to\infty} [H_{\mathrm{I}}, B^*(h_n) \otimes I]^0_{\mathcal{D}(H)} \Psi = 0.$$

For linear operator X, the spectrum of X is denoted by $\sigma(X)$ and that of essential spectrum by $\sigma_{\mathrm{ess}}(X)$. From **(A.1)**, it is seen that $\sigma(H) \subset \mathbf{R}$ and $E_0(H) > -\infty$ where $E_0(H) = \inf \sigma(H)$. The main result in this paper is as follows.

Theorem 1.1 *Assume* ***(A.1)*** *and* ***(A.2)****. Then*

$$\overline{\{E_0(H) + \lambda \mid \lambda \in \sigma_{ess}(K) \setminus \{0\}\}} \subset \sigma_{ess}(H), \tag{4}$$

where $\overline{J}$ denotes the closure of the subset $J \subset \mathbf{R}$.

The outline of the proof of Theorem 1.1 is as follows. Let X be a self-adjoint operator on a Hilbert space $\mathcal{X}$ and $\lambda \in \mathbf{R}$. Then a sequence $\{\psi_n\}_{n=1}^{\infty}$ of $\mathcal{X}$ is called the Weyl sequence for X and λ, if **(i)** $\psi_n \in \mathcal{D}(X)$ and $\|\Psi_n\| = 1$ for all $n \in \mathbf{N}$, **(ii)** w-$\lim\limits_{n\to\infty} \psi_n = 0$ and **(iii)** s-$\lim\limits_{n\to\infty}(X-\lambda)\psi_n = 0$. For a fundamental fact of the essential spectrum, there is a Weyl's criterion (see, e.g. [10]). Weyl's criterion says that $\lambda \in \sigma_{\mathrm{ess}}(X)$ if and only if there exists a Weyl sequence for X and λ. In a same strategy considered in [2], we directly construct the Weyl sequence for H and $E_0(H) + \lambda$ from the Weyl sequence for K and λ where $\lambda \in \sigma_{\mathrm{ess}}(K) \setminus \{0\}$.

This paper is organized as follows. In section 2, basic properties of Fock space and their operators are explained and the proof of the main theorem is given. In section 3, the application to the Yukawa model is considered.

2 Proof of Main Theorem

2.1 Fermion Fock Space and Boson Fock Space

In this subsection, basic properties of Fock spaces and their operators are given. The fermion Fock space on a Hilbert space $\mathcal{X}$ is defined by $\mathcal{F}_{\mathrm{f}}(\mathcal{X}) = \oplus_{n=0}^{\infty}(\otimes_a^n \mathcal{X})$ where $\otimes_a^n \mathcal{X}$ denotes the n-fold anti-symmetric tensor product of $\mathcal{X}$ with $\otimes_a^0 \mathcal{X} := \mathbf{C}$. The Fock vacuum is defined by $\Omega_{\mathrm{f}} = \{1, 0, 0, \cdots\} \in \mathcal{F}_{\mathrm{f}}(\mathcal{X})$. The annihilation operator is denoted by $B(f)$, $f \in \mathcal{X}$ and the creation operator by $B^*(g)$, $g \in \mathcal{X}$. For a subspace $\mathcal{M} \subset \mathcal{X}$, the finite particle subspace $\mathcal{F}_{\mathrm{f}}^{\mathrm{fin}}(\mathcal{M})$ is a set which consists of the vector $\Psi = B^*(f_1) \cdots B^*(f_n)\Omega_{\mathrm{f}}$, $f_j \in \mathcal{M}$, $j = 1, \cdots n$, $n \in \mathbf{N}$. It is known that $B(f)$ and $B^*(g)$ is bounded operator with

$$\|B(f)\| = \|f\|, \quad \|B^*(g)\| = \|g\|, \tag{5}$$

respectively, and they satisfy the canonical anti-commutation relations :

$$\{B(f), B^*(g)\} = (f, g)_{\mathcal{X}}, \tag{6}$$

$$\{B(f), B(g)\} = \{B^*(f), B^*(g)\} = 0, \tag{7}$$

where $\{X,Y\} = XY + YX$. Let X be a self-adjoint operator on $\mathcal{X}$ with bounded from below. The second quantization operator $d\Gamma_{\mathrm{f}}(X)$ is a self-adjoint operator on $\mathcal{F}_{\mathrm{f}}(\mathcal{X})$ and its acts for the finite particle vector $\Psi = B^*(f_1)\cdots B^*(f_n)\Omega_{\mathrm{f}}$ as $d\Gamma_{\mathrm{f}}(X)\Psi = \sum_{j=1}^{n} B^*(f_1)\cdots B^*(Xf_j)\cdots B^*(f_n)\Omega_{\mathrm{f}}$. For $f \in \mathcal{D}(X)$, it is seen that on $\mathcal{F}_{\mathrm{f}}^{\mathrm{fin}}(\mathcal{D}(X))$

$$[d\Gamma_{\mathrm{f}}(X), B(f)] = -d\Gamma_{\mathrm{f}}(Xf), \tag{8}$$

$$[d\Gamma_{\mathrm{f}}(X), B^*(f)] = d\Gamma_{\mathrm{f}}(Xf). \tag{9}$$

The boson Fock space on a Hilbert space $\mathcal{Y}$ is defined by $\mathcal{F}_{\mathrm{b}}(\mathcal{Y}) = \oplus_{n=0}^{\infty}(\otimes_s^n \mathcal{Y})$ where $\otimes_s^n \mathcal{Y}$ denotes the n-fold symmetric tensor product of $\mathcal{Y}$ with $\otimes_{\mathrm{s}}^0 \mathcal{Y} := \mathbf{C}$. The Fock vacuum is defined by $\Omega_{\mathrm{b}} = \{1, 0, 0, \cdots\} \in \mathcal{F}_{\mathrm{b}}(\mathcal{Y})$. The annihilation operator is denoted by $A(f)$, $f \in \mathcal{Y}$ and the creation operator by $A^*(g)$, $g \in \mathcal{Y}$. For a subspace $\mathcal{N} \subset \mathcal{Y}$, the finite particle subspace $\mathcal{F}_{\mathrm{b}}^{\mathrm{fin}}(\mathcal{N})$ is a set which consists of the vector $\Psi = A^*(f_1)\cdots A^*(f_n)\Omega_{\mathrm{b}}$, $f_j \in \mathcal{N}$, $j = 1, \cdots n$, $n \in \mathbf{N}$. Creation operators and annihilation of bosonic field satisfy the canonical commutation relations on the finite particle subspace $\mathcal{F}_{\mathrm{b}}^{\mathrm{fin}}(\mathcal{Y})$:

$$[A(f), A^*(g)] = (f, g)_{\mathcal{Y}}, \tag{10}$$

$$[A(f), A(g)] = [A^*(f), A^*(g)] = 0. \tag{11}$$

where $[X,Y] = XY - YX$. Let Y be a self-adjoint operator on $\mathcal{Y}$ with bounded from below . The second quantization operator $d\Gamma_{\mathrm{b}}(Y)$ is a self-adjoint on $\mathcal{F}_{\mathrm{b}}(Y)$ which act for the finite particle vector $\Psi = A^*(f_1)\cdots A^*(f_n)\Omega_{\mathrm{b}}$ as $d\Gamma_{\mathrm{b}}(Y)\Psi = \sum_{j=1}^{n} A^*(f_1)\cdots A^*(Yf_j)\cdots A^*(f_n)\Omega_{\mathrm{b}}$. For $f \in \mathcal{D}(Y)$, it follows that on $\mathcal{F}_{\mathrm{b}}^{\mathrm{fin}}(\mathcal{D}(Y))$,

$$[d\Gamma_{\mathrm{b}}(Y), A(f)] = -d\Gamma_{\mathrm{b}}(Yf), \tag{12}$$

$$[d\Gamma_{\mathrm{b}}(Y), A^*(f)] = d\Gamma_{\mathrm{b}}(Yf). \tag{13}$$

Let $f \in \mathcal{D}(Y^{-1/2})$. It follows that for $\Psi \in \mathcal{D}(d\Gamma_{\mathrm{b}}(Y)^{1/2})$,

$$\|A(f)\Psi\| \leq \|Y^{-1/2}f\| \, \|d\Gamma_{\mathrm{b}}(X)^{1/2}\Psi\|, \tag{14}$$

$$\|A^*(f)\Psi\| \leq \|Y^{-1/2}f\| \, \|d\Gamma_{\mathrm{b}}(Y)^{1/2}\Psi\| + \|f\|\|\Psi\|. \tag{15}$$

2.2 Proof of Thorem1.1

Lemma 2.1 *Let $\{h_n\}_{n=1}^{\infty}$ be a sequence of $\mathcal{K}$ satisfying $w\text{-}\lim_{n\to\infty} h_n = 0$. Then for $\Psi \in \mathcal{F}_f(\mathcal{K})$,*

(1) $s\text{-}\lim_{n\to\infty} B(h_n)\Psi = 0,$

(2) $w\text{-}\lim_{n\to\infty} B^*(h_n)\Psi = 0.$

(Proof)
(1) Let $\Psi = B^*(f_1)\cdots B^*(f_l)\Omega_{\mathrm{f}} \in \mathcal{F}_{\mathrm{f}}^{\mathrm{fin}}(\mathcal{K})$ be a finite particle vector. From canonical anti-commutation relations (6) and (7), it is seen that

$$B(h_n)\Psi = \sum_{j=1}^{l}(-1)^j(h_n, f_j)B^*(f_1)\cdots\widehat{B^*(f_j)}\cdots B^*(f_l)\Omega_{\mathrm{f}}$$

where $\widehat{X}$ stands for omitting the operator X. Since w-$\lim\limits_{n\to\infty} h_n = 0$, it follows that $\lim\limits_{n\to\infty}\|B(h_n)\Psi\| = 0$. Then we see that for all finite vector $\Psi \in \mathcal{F}_{\mathrm{f}}^{\mathrm{fin}}(\mathcal{K})$, s-$\lim\limits_{n\to\infty} b(h_n)\Psi = 0$. Note that $\mathcal{F}_{\mathrm{f}}^{\mathrm{fin}}(\mathcal{K})$ is dense in $\mathcal{F}_{\mathrm{f}}(\mathcal{K})$ and the value of the norm $\|B(h_n)\|$ is uniformly bounded with $\|B(h_n)\| = \|h_n\| = 1$, for all $n \in \mathbf{N}$. Then we see that s-$\lim\limits_{n\to\infty} B(h_n)\Psi = 0$ for all $\Psi \in \mathcal{F}_{\mathrm{f}}(\mathcal{K})$.
(2) Let $\Psi \in \mathcal{F}_{\mathrm{f}}(\mathcal{K})$. By using canonical anti-commutation relations (6) and (7), we see that for $\Phi = B^*(g_1)\cdots B^*(g_l)\Omega_{\mathrm{f}} \in \mathcal{F}_{\mathrm{f}}^{\mathrm{fin}}(\mathcal{K})$,

$$(\Phi, B^*(h_n)\Psi) = (B(h_n)\Phi, \Psi) = \sum_{j=1}^{l}(-1)^j(g_j, h_n)(B^*(g_1)\cdots\widehat{B^*(g_j)}\cdots B^*(g_l)\Omega_{\mathrm{f}}, \Psi).$$

From this equality and w-$\lim\limits_{n\to\infty} h_n = 0$, we have $\lim\limits_{n\to\infty}(\Phi, B^*(h_n)\Psi) = 0$. Note that $\mathcal{F}_{\mathrm{f}}^{\mathrm{fin}}(\mathcal{K})$ is dense in $\mathcal{F}_{\mathrm{f}}(\mathcal{K})$ and $\|b^*(h_n)\| = \|h_n\| = 1$ for all $n \in \mathbf{N}$. Hence we see that for all $\Phi \in \mathcal{F}_{\mathrm{f}}(\mathcal{K})$, $\lim\limits_{n\to\infty}(\Phi, B^*(h_n)\Psi) = 0$. ■

Lemma 2.2 *It follows that for $f \in \mathcal{D}(K)$,*

$$\textbf{\textit{(i)}} \quad [H_0, B(f)\otimes I]^0_{\mathcal{D}(H_0)} = -B(Kf)\otimes I_{\restriction\mathcal{D}(H_0)},$$
$$\textbf{\textit{(ii)}} \quad [H_0, B^*(f)\otimes I]^0_{\mathcal{D}(H_0)} = B^*(Kf)\otimes I_{\restriction\mathcal{D}(H_0)}.$$

(Proof) From commutation relation (8), we see that for $\Psi \in \mathcal{F}_{\mathrm{f}}^{\mathrm{fin}}(\mathcal{D}(K)) \otimes_{\mathrm{alg}} \mathcal{D}(T)$ where $\otimes_{\mathrm{alg}}$ denotes the algebraic tensor product and $\Phi \in \mathcal{D}(H_0)$,

$$(H_0\Phi, (B(f)\otimes I)\Psi) - ((B^*(f)\otimes I)\Phi, H_0\Psi) = -(\Phi, (B(Kf)\otimes I)\Psi).$$

Note that $\mathcal{F}_{\mathrm{f}}^{\mathrm{fin}}(\mathcal{D}(K)) \otimes_{\mathrm{alg}} \mathcal{D}(T)$ is a core of H_0. Since $B(f)$ and $B^*(f)$ are bouded, we see that for all $\Psi \in \mathcal{D}(H_0)$, $(H_0\Phi, (B(f)\otimes I)\Psi) - ((B^*(f)\otimes I)\Phi, H_0\Psi) = -(\Phi, (B(Kf)\otimes I)\Psi)$. Thus **(i)** holds. Similarly, we can obtain **(ii)**. ■

(Proof of Theorem 1.1)

Let $\lambda \in \sigma_{\mathrm{ess}}(K)\backslash\{0\}$. Then there exists Weyl sequence $\{h_n\}_{n=1}^{\infty}$ for K and λ i.e. $h_n \in \mathcal{D}(K)$ and $\|h_n\| = 1$ for $n \in \mathbf{N}$, w-$\lim\limits_{n\to\infty} h_n = 0$, s-$\lim\limits_{n\to\infty}(K-\lambda)h_n = 0$. By using this sequence, we construct the Weyl sequence for for H and $E_0(H)+\lambda$. Let us set

$$\Psi_{n,\varepsilon} \ = \ ((B(h_n) + B^*(h_n))\otimes I)\,\Xi_\varepsilon, \tag{16}$$

where $\Xi_\varepsilon \in \mathrm{Ran}(E_H([0,\varepsilon)))$, $\|\Xi_\varepsilon\| = 1$, $0 < \varepsilon \leq 1$. Here E_H denotes the spectral projection of H. From canonical anti-commutation relations (6) and (7), we see that

$$\begin{aligned}\|\Psi_{n,\varepsilon}\|^2 &= (\Xi_\varepsilon, \big((B(h_n)^2 + \{B(h_n), B^*(h_n)\} + B^*(h_n)^2) \otimes I\big)\, \Xi_\varepsilon) \\ &= \|h_n\|^2\, \|\Xi_\varepsilon\|^2. \end{aligned} \tag{17}$$

Since $\|h_n\| = 1$ and $\|\Xi_\varepsilon\| = 1$, we see that $\|\Psi_{n,\varepsilon}\| = 1$ for all $n \geq 1$ and $0 < \varepsilon \leq 1$. From Lemma 2.2 and the assumption **(A.2)**, it is seen that for $\Phi, \Theta \in \mathcal{D}(H)$ and $h \in \mathcal{D}(K)$,

$$\begin{aligned}&(H\Phi, ((B(h) + B^*(h)) \otimes I)\Theta) - (((B(h) + B^*(h)) \otimes I)\Phi, H\Theta) \\ &\quad = \left(\Phi, \left((B^*(Kh) - B(Kh)) \otimes I + [H_{\mathrm{I}}, B^*(h) \otimes I]^0_{\mathcal{D}(H)} + [H_{\mathrm{I}}, B(h) \otimes I]^0_{\mathcal{D}(H)}\right) \Theta\right).\end{aligned}$$

Hence we have for $\Phi \in \mathcal{D}(H)$,

$$\begin{aligned}(H\Phi, \Psi_{n,\varepsilon}) &= (\Phi, ((B(h_n) + B^*(h_n)) \otimes I) H \Xi_\varepsilon) + (\Phi, ((B^*(Kh_n) - B(Kh_n)) \otimes I)\Xi_\varepsilon) \\ &\quad + (\Phi, [H_{\mathrm{I}}, (B^*(h_n) \otimes I)]^0_{\mathcal{D}(H)} \Xi_\varepsilon) + (\Phi, [H_{\mathrm{I}}, (B(h_n) \otimes I)]^0_{\mathcal{D}(H)} \Xi_\varepsilon).\end{aligned} \tag{18}$$

Then we see that $\Psi_{n,\varepsilon} \in \mathcal{D}(H^*)$ and hence $\Psi \in \mathcal{D}(H)$ since H is self-adjoint. Then we have

$$\begin{aligned}H\Psi_{n,\varepsilon} =& ((B(h_n) + B^*(h_n)) \otimes I) H \Xi_\varepsilon + ((B^*(Kh_n) - B(Kh_n)) \otimes I)\Xi_\varepsilon \\ &+ [H_{\mathrm{I}}, B^*(h_n) \otimes I]^0_{\mathcal{D}(H)} \Xi_\varepsilon + [H_{\mathrm{I}}, B(h_n) \otimes I]^0_{\mathcal{D}(H)} \Xi_\varepsilon.\end{aligned} \tag{19}$$

From this equality, we have

$$\begin{aligned}&\|\,(H - (\lambda + E_0(H)))\,\Psi_{n,\varepsilon}\| \\ &\quad \leq \|((B(h_n) + B^*(h_n)) \otimes I)(H - E_0(H))\Xi_\varepsilon\| + \|(B^*(Kh_n - \lambda h_n) \otimes I)\Xi_\varepsilon\| \\ &\qquad + \|B(Kh_n + \lambda h_n)\Xi_\varepsilon\| + \|[H, B(h_n) \otimes I]^0_{\mathcal{D}(H)} \Xi_\varepsilon\| + \|[H, B^*(h_n) \otimes I]^0_{\mathcal{D}(H)} \Xi_\varepsilon\|.\end{aligned} \tag{20}$$

We see that

$$\|((B(h_n) + B^*(h_n)) \otimes I)(H - E_0(H))\Xi_\varepsilon\| \leq 2\|h_n\|\|(H - E_0(H))\Xi_\varepsilon\| \leq 2\varepsilon. \tag{21}$$

We also see that

$$\|(B^*(Kh_n - \lambda h_n) \otimes I)\Xi_\varepsilon\| \leq \|(K - \lambda)h_n\|, \tag{22}$$

and using $B(K+\lambda)h_n = B(K-\lambda)h_n + 2\lambda B(h_n)$, we have

$$\|(B((K+\lambda)h_n) \otimes I)\Xi_\varepsilon\| \leq \|(K-\lambda)h_n\| + 2|\lambda|\|(B(h_n) \otimes I)\Xi_\varepsilon\|. \tag{23}$$

By applying (21), (22) and (23) to (20),

$$\|\,(H - (\lambda + E_0(H)))\,\Psi_{n,\varepsilon}\| \leq 2\varepsilon + 2\|(K-\lambda)h_n\| + 2|\lambda|\|(B(h_n) \otimes I)\Xi_\varepsilon\|. \tag{24}$$

Here it is noted that s-$\lim_{n\to\infty}(B(h_n)\otimes I)\Xi_\varepsilon = 0$ by Lemma 2.1. From this fact and s-$\lim_{n\to\infty}(K-\lambda)h_n = 0$, (24) yields that

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\|\,(H-(\lambda+E_0(H)))\,\Psi_{n,\varepsilon}\| = 0.$$

Then, we can take a subsequence $\{\Psi_{n_j,\varepsilon_j}\}_{j=1}^{\infty}$ satisfying $\lim_{j\to\infty}\|\,(H-(\lambda+E_0(H)))\,\Psi_{n_j,\varepsilon_j}\| = 0$. In addition, from the definition of $\Psi_{n,j}$ and Lemma 2.1, we see that w-$\lim_{j\to\infty}\Psi_{n_j,\varepsilon_j} = 0$. Then we have $E_0(H)+\lambda \in \sigma_{\rm ess}(H)$ for $\lambda \in \sigma_{\rm ess}(K)\backslash\{0\}$. Since $\sigma_{\rm ess}(H)$ is closed subset of $\mathbf{R}$, the proof is obtained. ■

3 Application

In this subsection, the application of the main theorem to Yukawa model is consided. Yukawa model describes the system of Dirac fields coupled to Klein-Gordon fields. The state space of Dirac field and Klein-Gordon field are given by $\mathcal{H}_{\rm D} = \mathcal{F}_{\rm f}(L^2(\mathbf{R}^3_{\mathbf{p}};\mathbf{C}^4))$ and $\mathcal{H}_{\rm KG} = \mathcal{F}_{\rm b}(L^2(\mathbf{R}^3_{\mathbf{k}}))$, respectively. The state space of Yukawa model is defined by

$$\mathcal{H}_{\rm Y} = \mathcal{H}_{\rm D}\otimes\mathcal{H}_{\rm KG}$$

and the total Hamiltonian by

$$H_\kappa = H_{\rm D}\otimes I + I\otimes H_{\rm KG} + \kappa H_{\rm I}, \qquad \kappa\in\mathbf{R},$$

where $H_{\rm D}$ and $H_{\rm KG}$ are the free Hamiltonians of the Dirac field and the Klein-Gordon field that are defined by $H_{\rm D} = d\Gamma_{\rm f}(\omega_M)$ with $\omega_M(\mathbf{p}) = \sqrt{\mathbf{p}^2+M^2}$, $M>0$, and $H_{\rm KG} = d\Gamma_{\rm b}(\omega_m)$ with $\omega_m(\mathbf{k}) = \sqrt{\mathbf{k}^2+m^2}$, $m>0$, respectively, and $H_{\rm I}$ is a the interaction which is a symmetric operator on $\mathcal{H}_{\rm Y}$ satisfying for $\Phi\in\mathcal{H}_{\rm Y}$ and $\Psi\in\mathcal{D}(H_0)$,

$$(\Phi, H_{\rm I}\Psi) = \int_{\mathbf{R}^3}\chi_{\rm I}(\mathbf{x})\left(\Phi,\,\overline{\psi(\mathbf{x})}\psi(\mathbf{x})\otimes\phi(\mathbf{x})\Psi\right)d\mathbf{x}, \tag{25}$$

where $\psi(\mathbf{x})$ and $\phi(\mathbf{x})$ are field operators of the Dirac field and Klein-Gordon field, respectively, and $\overline{\psi(\mathbf{x})} = \psi^*(\mathbf{x})\beta$. In this paper, we use the Dirac matrices α^j, $j=1,2,3$, and β that are the 4×4 hermitian matrix satisfying the anti-commutation relation $\{\alpha^j,\alpha^l\} = 2\delta_{j,l}$, $\{\alpha_j,\beta\} = 0$, $\beta^2 = I$. The definitions of $\psi(\mathbf{x})$ and $\phi(\mathbf{x})$ are as follows. First we consider the dirac field's operator. Let $B(\xi)$, $\xi = (\xi_1,\cdots,\xi_4)\in\mathcal{H}_{\rm D}$, and $B^*(\eta)$, $\eta = (\eta_1,\cdots,\eta_4)\in\mathcal{H}_{\rm D}$, be the annihilation operator and the creation operator on $\mathcal{H}_{\rm D}$, respectively. For $f\in L^2(\mathbf{R}^3)$ let us set

$$b^*_{1/2}(f) = B^*((f,0,0,0)), \qquad b^*_{-1/2}(f) = B^*((0,f,0,0)),$$
$$d^*_{1/2}(f) = B^*((0,0,f,0)), \qquad d^*_{-1/2}(f) = B^*((0,0,0,f)).$$

Then the next canonical anti-commutation relations follow :

$$\{b_s(f), b_\tau^*(g)\} = \{d_s(f), d_\tau^*(g)\} = \delta_{s,\tau}(f,g)_{L^2(\mathbf{R}^3)}, \tag{26}$$

$$\{b_s(f), b_\tau(g)\} = \{d_s(f), d_\tau(g)\} = \{b_s(f), d_\tau(g)\} = \{b_s(f), d_\tau^*(g)\} = 0. \tag{27}$$

Let $u_s(\mathbf{p}) = (u_s^l(\mathbf{p}))_{l=1}^4$ and $v_s(\mathbf{p}) = (v_s^l(\mathbf{p}))_{l=1}^4$ be spinors with spin $s = \pm 1/2$ that are the positive and negative energy part of the Fourier transformed Dirac operator $h_{\mathrm{D}}(\mathbf{p}) = \alpha \cdot \mathbf{p} + \beta M$, respectively (see, e.g. [13]). Then the filed operator $\psi(\mathbf{x}) = (\psi_l(\mathbf{x}))_{l=1}^4$ of the Dirac field is defined by

$$\psi_l(\mathbf{x}) = \sum_{s=\pm 1/2} (b_s(f_{s,\mathbf{x}}^l) + d_s^*(g_{s,\mathbf{x}}^l)), \quad l = 1, \cdots, 4,$$

where $f_{s,\mathbf{x}}^l(\mathbf{p}) = f_s^l(\mathbf{p}) e^{-i\mathbf{p}\cdot\mathbf{x}}$ with $f_s^l(\mathbf{p}) = \frac{\chi_{\mathrm{D}}(\mathbf{p}) u_s^l(\mathbf{p})}{\sqrt{(2\pi)^3 \omega_M(\mathbf{p})}}$ and $g_{s,\mathbf{x}}^l(\mathbf{p}) = g_s^l(\mathbf{p}) e^{-i\mathbf{p}\cdot\mathbf{x}}$ with $g_s^l(\mathbf{p}) = \frac{\chi_{\mathrm{D}}(\mathbf{p}) v_s^l(-\mathbf{p})}{\sqrt{(2\pi)^3 \omega_M(\mathbf{p})}}$. Here χ_{D} denotes the ultraviolet cutoff. Next we define the Klein-Gordon filed's operator. Let $a(f)$, $f \in \mathcal{H}_{\mathrm{KG}}$ and $a^*(g), g \in \mathcal{H}_{\mathrm{KG}}$ be the annihilation operator and creation operator of the Klein-Gordon field. Then it is seen that on $\mathcal{F}_{\mathrm{b}}^{\mathrm{fin}}(\mathcal{H}_{\mathrm{KG}})$,

$$[a(f), a^*(g)] = 0 \qquad [a(f), a(g)] = [a^*(f), a^*(g)] = 0. \tag{28}$$

The field operator of the Klein-Gordon field is defined by

$$\phi(\mathbf{x}) \ = \ \frac{1}{\sqrt{2}} \Big(\, a(h_{\mathbf{x}}) \ + \ a^*(h_{\mathbf{x}}) \Big) \, ,$$

where $h_{\mathbf{x}}(\mathbf{k}) = h(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}$ with $h(\mathbf{k}) \ = \ \frac{\chi_{\mathrm{KG}}(\mathbf{k})}{\sqrt{(2\pi)^3 \omega_m(\mathbf{k})}}$, and χ_{KG} is the ultraviolet cutoff.

We suppose the following conditions

(Y.1) (Ultraviolet cutoff)

$$\int_{\mathbf{R}^3} \left| \chi_{\mathrm{D}}(\mathbf{p}) u_s^l(\mathbf{p}) \right|^2 d\mathbf{p} \ < \ \infty, \left| \chi_{\mathrm{D}}(\mathbf{p}) v_s^l(-\mathbf{p}) \right|^2 d\mathbf{p} \ < \ \infty, \int_{\mathbf{R}^3} |\chi_{\mathrm{KG}}(\mathbf{k})|^2 d\mathbf{k} \ < \ \infty.$$

(Y.2) (Spatial cutoff) $\int_{\mathbf{R}^3} |\chi_{\mathrm{I}}(\mathbf{x})| d\mathbf{x} < \infty$.

From the boundness (5), (14), (15) and $\overline{\psi(\mathbf{x})}\psi(\mathbf{x}) = \sum_{l,l'} \beta_{l,l'} \psi_l^*(\mathbf{x}) \psi_{l'}(\mathbf{x})$, we see by **(Y.1)** that

$$\sup_{\mathbf{x}\in\mathbf{R}^3} \| \overline{\psi(\mathbf{x})} \psi(\mathbf{x}) \| \leq \sum_{l,l'=1}^{4} \sum_{s,s'=\pm 1/2} |\beta_{l,l'}| (\| f_s^l \| + \| g_s^l \|)(\| f_{s'}^l \| + \| g_{s'}^l \|), \tag{29}$$

$$\sup_{\mathbf{x}\in\mathbf{R}^3} \| \phi(\mathbf{x}) \Psi \| \leq \sqrt{2} \| \frac{h}{\sqrt{\omega_m}} \| \, \| H_{\mathrm{KG}}^{1/2} \Psi \| + \frac{1}{\sqrt{2}} \| h \| \| \Psi \|. \tag{30}$$

Then from (29), (30), **(Y.2)** and $\|H_{\rm KG}^{1/2}\Phi\| \leq \varepsilon\|H_{\rm KG}\Phi\| + \frac{1}{2\varepsilon}\|\Phi\|$, it is seen that for $\Psi \in \mathcal{D}(H_0)$,

$$\|H_{\rm I}\Psi\| \leq \varepsilon\sqrt{2}\|\chi_{\rm I}\|_{L^1}\|\frac{h}{\omega_{\rm KG}}\|\|H_0\Psi\| + \|\chi_{\rm I}\|_{L^1}\left(\frac{1}{\sqrt{2}\varepsilon}\|\frac{h}{\omega_m}\| + \frac{1}{\sqrt{2}}\|h\| \right.$$
$$\left. + \sum_{l,l'=1}^{4}\sum_{s,s'=\pm 1/2}|\beta_{l,l'}|(\|f_s^l\| + \|g_s^l\|)(\|f_{s'}^l\| + \|g_{s'}^l\|)\right)\|\Psi\|.$$

Thus the interaction is relatively bounded with respect to H_0 with infinitely small bound, and then Kato-Rellich theorem shows that H_κ is self-adjoint and essentially self-adjoint on any core of H_0. In particular H_κ is essentially self-adjoint on $\mathcal{F}_{\rm f}^{\rm fin}(\mathcal{D}(\omega_{\rm D}))\otimes_{\rm alg}\mathcal{F}_{\rm b}^{\rm fin}(\mathcal{D}(\omega_{\rm KG}))$ where $\otimes_{\rm alg}$ stands for the algebraic tensor product.

Let $\nu = \min\{m, M\}$. From Theorem1.1, the next assertion follows.

Theorem 3.1
*Assume **(Y.1) - (Y.2)**. Then* $[E_0(H_\kappa)+\nu,\infty) \subset \sigma_{ess}(H_\kappa)$ *for all* $\kappa \in \mathbf{R}$.

Before starting the proof of the Theorem 3.1, we explain a result of the essential spectrum of Yukawa model and state a corollary. In [12], the following theorem has been proven:

> **Theorem ([12] ; Theorem 2.1)**
> Assume **(Y.1)**, **(Y.2)** and $\int_{\mathbf{R}^3}|\mathbf{x}|\,|\chi_{\rm I}(\mathbf{x})|d\mathbf{x} < \infty$. Then it follows that for all $\kappa \in \mathbf{R}$, $[E_0(H_\kappa)+\nu,\infty) \subset \sigma_{\rm ess}(H_\kappa)$.

Then from this result and Theorem 3.1, next corollary follows.

Corollary 3.2 ***(HVZ theorem for Yukawa model)***
*Assume **(Y.1), (Y.2)** and* $\int_{\mathbf{R}^3}|\mathbf{x}|\,|\chi_I(\mathbf{x})|d\mathbf{x} < \infty$. *Then* $[E_0(H_\kappa)+\nu,\infty) = \sigma_{ess}(H_\kappa)$ *for all* $\kappa \in \mathbf{R}$.

To prove the Theorem 3.1, we prepare for some lemmas.

Lemma 3.3 *Let A and B be self-adjoint operators on Hilbert spaces $\mathcal{X}$ and $\mathcal{Y}$, respectively. Assume that A and B are non-negative. Let $X(\mathbf{x})$ be a linear operator on $\mathcal{F}_f(\mathcal{X})$ for $\mathbf{x} \in \mathbf{R}^d$, and $Y(\mathbf{x})$ a linear operator on $\mathcal{F}_b(\mathcal{Y})$ for $\mathbf{y} \in \mathbf{R}^d$ that satisfy*

$$\sup_{\mathbf{x}\in\mathbf{R}^d}\|X(\mathbf{x})\Psi\| \leq const.(\|d\Gamma_f(A)^{1/2}\Psi\| + \|\Psi\|), \quad \Psi \in \mathcal{D}(d\Gamma_f(A)^{1/2}), \tag{31}$$

$$\sup_{\mathbf{x}\in\mathbf{R}^d}\|Y(\mathbf{x})\Psi\| \leq const.(\|d\Gamma_b(B)^{1/2}\Psi\| + \|\Psi\|), \quad \Psi \in \mathcal{D}(d\Gamma_b(B)^{1/2}), \tag{32}$$

respectively. Then there exists an linear operator Z on $\mathcal{F}_f(\mathcal{X})\otimes\mathcal{F}_b(\mathcal{Y})$ satisfying $\mathcal{D}(d\Gamma_f(A)\otimes I)\cap\mathcal{D}(I\otimes d\Gamma_b(B)) \subset \mathcal{D}(Z)$ and for $\Phi \in \mathcal{F}_f(\mathcal{X})\otimes\mathcal{F}_b(\mathcal{Y})$ and $\Psi \in \mathcal{D}(d\Gamma_f(A)\otimes I)\cap\mathcal{D}(I\otimes d\Gamma_b(B))$,

$$(\Phi, Z\Psi) = \int_{\mathbf{R}^d} g(\mathbf{x})(\Phi,(X(\mathbf{x})\otimes Y(\mathbf{x})\Psi)d\mathbf{x},$$

where g is the Borel function on $\mathbf{R}^d$ satisfying $\|g\|_{L^1} < \infty$.

(Proof) Let us set a linear functional ℓ on $(\mathcal{F}_{\mathrm{f}}(\mathcal{X}) \otimes \mathcal{F}_{\mathrm{b}}(\mathcal{Y})) \times (\mathcal{F}_{\mathrm{f}}(\mathcal{D}(A)) \otimes \mathcal{F}_{\mathrm{b}}(\mathcal{D}(B)))$ by

$$\ell(\Phi, \Psi) \; = \; \int_{\mathbf{R}^d} g(\mathbf{x})(\Phi, (X(\mathbf{x}) \otimes Y(\mathbf{x})\Psi) d\mathbf{x}. \tag{33}$$

It is seen that $|\ell(\Phi, \Psi)| \leq c \|g\|_{L^1} \|\Phi\|(\|(d\Gamma_{\mathrm{f}}(A) \otimes I + I \otimes d\Gamma_{\mathrm{b}}(B)\Psi\| + \|\Psi\|)$ with some constant $c > 0$. Then from Riez representation theorem, the assersion holds. ■

By using canonical anti-commutation relations (26) and (27), it has been proven in Lemma ([11] ; Lemma 3.1) that the next commutation relations follow :

$$[\psi_l^*(\mathbf{x})\, \psi_{l'}(\mathbf{x}),\; b_s(\xi)] = -(\xi, f_{s,\mathbf{x}}^l)\, \psi_{l'}(\mathbf{x}), \tag{34}$$

$$[\psi_l^*(\mathbf{x})\, \psi_{l'}(\mathbf{x}),\; d_s(\xi)] = (\xi, g_{s,\mathbf{x}}^{l'})\, \psi_l^*(\mathbf{x}). \tag{35}$$

By (39), (40), and the operators equality $[X,Y]^* = -[X^*, Y^*]$, it is seen that

$$[\psi_r(\mathbf{x})\, \psi_{r'}^*(\mathbf{x}),\; b_s^*(\eta)] = (f_{s,\mathbf{x}}^{r'}, \eta)\, \psi_r^*(\mathbf{x}), \tag{36}$$

$$[\psi_r(\mathbf{x})\, \psi_{r'}^*(\mathbf{x}),\; d_s^*(\eta)] = -(g_{s,\mathbf{x}}^{r}, \eta)\, \psi_{r'}(\mathbf{x}). \tag{37}$$

Remark 3.1 *For the notation of* $\psi(\mathbf{x}) = (\psi_l(\mathbf{x}))_{l=1}^4$ *in the paper [11], it is defined by* $\psi_l(\mathbf{x}) = \sum_{s=\pm 1/2} (b_s(g_{s,\mathbf{x}}^l) + d_s^*(h_{s,\mathbf{x}}^l))$ *where* $g_{s,\mathbf{x}}^l(\mathbf{p}) = \frac{\chi_D(\mathbf{p}) u_s^l(\mathbf{p})}{\sqrt{(2\pi)^3 \omega_M(\mathbf{p})}} e^{-i\mathbf{p}\cdot\mathbf{x}}$ *and* $h_{s,\mathbf{x}}^l(\mathbf{p}) = \frac{\chi_D(\mathbf{p}) v_s^l(-\mathbf{p})}{\sqrt{(2\pi)^3 \omega_M(\mathbf{p})}} e^{i\mathbf{p}\cdot\mathbf{x}}, s = \pm 1/2,\ l = 1, \cdots, 4.$

Here note that

$$\sup_{\mathbf{x}\in\mathbf{R}^3} \|\psi_l(\mathbf{x})\| \leq \sum_{s=\pm 1/2} (\|f_s^l\| + \|g_s^l\|). \tag{38}$$

By applying the commutation relations (34) - (37) and the norm-boundness (30) and (38) to the Lemma 3.3, the next lemma follows.

Lemma 3.4
(1)*There exits* $[H_I, b_s(\xi) \otimes I]^0_{\mathcal{D}(H)}$ *and* $[H_I, d_s(\xi) \otimes I]^0_{\mathcal{D}(H)}$ *satisfying for* $\Phi \in H_Y$ *and* $\Psi \in \mathcal{D}(H_0)$*,*

$$(\Phi, [H_I, b_s(\xi) \otimes I]^0_{\mathcal{D}(H)} \Psi) \; = \; -\sum_{l,l'} \beta_{l,l'} \int_{\mathbf{R}^3} \chi_I(\mathbf{x}) (\xi, f_{s,\mathbf{x}}^l)\, (\Phi, (\psi_{l'}(\mathbf{x}) \otimes \phi(\mathbf{x}))\Psi)\, d\mathbf{x}, \tag{39}$$

$$(\Phi, [H_I, d_s(\xi) \otimes I]^0_{\mathcal{D}(H)} \Psi) \; = \; \sum_{l,l'} \beta_{l,l'} \int_{\mathbf{R}^3} \chi_I(\mathbf{x}) (\xi, g_{s,\mathbf{x}}^{l'})\, (\Phi, \psi_l^*(\mathbf{x}) \otimes \phi(\mathbf{x}))\Psi)\, d\mathbf{x}. \tag{40}$$

(2) There exits $[H_I, b_s^*(\eta)\otimes I]^0_{\mathcal{D}(H)}$ *and* $[H_I, d_s^*(\eta)\otimes I]^0_{\mathcal{D}(H)}$ *satisfying for* $\Phi \in H_Y$ *and* $\Psi \in \mathcal{D}(H_0)$

$$(\Phi, [H_I, b_s^*(\eta)\otimes I]^0_{\mathcal{D}(H)}\Psi) \;=\; \sum_{r,r'}\beta_{r,r'}\int_{\mathbf{R}^3}\chi_I(\mathbf{x})(f^{r'}_{s,\mathbf{x}},\eta)\,(\Phi,(\psi_r^*(\mathbf{x})\otimes\phi(\mathbf{x}))\Psi)\,d\mathbf{x}, \tag{41}$$

$$(\Phi, [H_I, d_s^*(\eta)\otimes I]^0_{\mathcal{D}(H)}\Psi) \;=\; -\sum_{r,r'}\beta_{r,r'}\int_{\mathbf{R}^3}\chi_I(\mathbf{x})(g^{r}_{s,\mathbf{x}},\eta)\,(\Phi(\psi_{r'}(\mathbf{x})\otimes\phi(\mathbf{x}))\Psi)\,d\mathbf{x}. \tag{42}$$

Similarly, applying (28) and (29) to Lemma 3.3, the subsequent proposition follows :

Lemma 3.5 *There exits* $[H_I, I\otimes a^*(\zeta)]^0_{\mathcal{D}(H)}$ *satisfying for* $\Phi \in H_Y$ *and* $\Psi \in \mathcal{D}(H_0)$*,*

$$(\Phi, [H_I, I\otimes a^*(\zeta)]^0_{\mathcal{D}(H)}\Psi) \;=\; \int_{\mathbf{R}^3}\chi_I(\mathbf{x})(f_{\mathbf{x}},\zeta)(\Phi,(\overline{\psi(\mathbf{x})}\psi(\mathbf{x})\otimes I)\Psi)d\mathbf{x}.$$

(Proof of Theorem 3.1)

Let us apply to the Theorem 1.1 to Yukawa model. Since H_κ is self-adjoint on $\mathcal{D}(H_0)$ with bounded from below, **(A.1)** is satisfied. Let us check the condition **(A.2)**. Let $\{h_n\}_{n=1}^\infty$ be the sequence satisfying $h_n \in \mathcal{D}(\omega_M)$, $\|h_n\| = 1$ for all $n \in \mathbf{N}$, and $\text{w-}\lim_{n\to\infty} h_n = 0$. From (39), it is seen that for all $\Psi \in \mathcal{D}(H)$,

$$\|[H_{\mathrm{I}}, b_s(h_n)\otimes I]^0_{\mathcal{D}(H)}\Psi\| \;\le \sum_{l,l'}|\beta_{l,l'}|\int_{\mathbf{R}^3}\chi_{\mathrm{I}}(\mathbf{x})|(h_n, f^l_{s,\mathbf{x}})|\,\|(\psi_{l'}(\mathbf{x})\otimes\phi(\mathbf{x}))\Psi\|d\mathbf{x}. \tag{43}$$

By using (38) and (30), we have $\sup_{\mathbf{x}\in\mathbf{R}^3}\|(\psi_{l'}(\mathbf{x})\otimes\phi(\mathbf{x}))\Psi\| < \infty$. We also see that $\int_{\mathbf{R}^3}|\chi_{\mathrm{I}}(\mathbf{x})|d\mathbf{x} < \infty$ by **(A.1)** and $|(h_n, f^l_{s,\mathbf{x}})| \le \|f_s\|$. Then from $\text{w-}\lim_{n\to\infty} h_n = 0$ and Lebesgue dominated convergence theorem, (43) yields that $\lim_{n\to\infty}\|[H_{\mathrm{I}}, b_s(h_n)\otimes I]^0_{\mathcal{D}(H)}\Psi\| = 0$. Similarly we can prove that $\lim_{n\to\infty}\|[H_{\mathrm{I}}, d_s(h_n)\otimes I]^0_{\mathcal{D}(H)}\Psi\| = 0$. Then the condition (1) in **(A.2)** is satisfied. In addition, we can also prove that $\lim_{n\to\infty}\|[H_{\mathrm{I}}, b_s^*(h_n)\otimes I]^0_{\mathcal{D}(H)}\Psi\| = 0$ and $\lim_{n\to\infty}\|[H_{\mathrm{I}}, d_s^*(h_n)\otimes I]^0_{\mathcal{D}(H)}\Psi\| = 0$, and then the condition (2) in **(A.2)** is satisfied. Hence from Theorem 1.1, it follows that $[E_0(H_\kappa)+M, \infty) \subset \sigma_{\mathrm{ess}}(H_\kappa)$. Next we show $[E_0(H_\kappa)+m, \infty) \subset \sigma_{\mathrm{ess}}(H_\kappa)$. Then we have $[E_0(H_\kappa)+\nu, \infty) \subset \sigma_{\mathrm{ess}}(H_\kappa)$ and Theorem 3.1 follows. Here note that ω_m^{-s} is bounded for $s > 0$ since $\omega_m > 0$. Let $\{f_n\}_{n=1}^\infty$ be the sequence of $\mathcal{H}_m$ satisfying $f_n \in \mathcal{D}(\omega_m)$, $\|f_n\| = 1$ for all $n \in \mathbf{N}$, and $\text{w-}\lim_{n\to\infty} h_n = 0$. It is proven from canonical commutation relations (28) and Lemma 3.5 that

$$\|[H_{\mathrm{I}}, I\otimes a^*(f_n)]^0_{\mathcal{D}(H)}\Psi\| \;\le\; \int_{\mathbf{R}^3}|\chi_{\mathrm{I}}(\mathbf{x})|\,|(f_{\mathbf{x}}, f_n)|\|(\overline{\psi(\mathbf{x})}\psi(\mathbf{x})\otimes I)\Psi\|d\mathbf{x}\,. \tag{44}$$

We see that $\chi_I \in L^1$, $|(h_{\mathbf{x}}, f_n)| \le \|f\|$, $\text{w-}\lim_{n\to\infty} f_n = 0$ and $\sup_{\mathbf{x}\in\mathbf{R}^3}\|\overline{\psi(\mathbf{x})}\psi(\mathbf{x})\| < \infty$. Then applying Lebesgue dominated convergence theorem to (44), we have $\lim_{n\to\infty}\|[H_{\mathrm{I}}, I\otimes a^*(f_n)]^0_{\mathcal{D}(H)}\Psi\| \;= 0$

Then the condition **(S.2)** of Theorem A ([2] ; Theorem 1.2) in Appendix is satisfied and it follows that $[E_0(H_\kappa)+m,\infty) \subset \sigma_{\rm ess}(H_\kappa)$. ■

Appendix

Here we review the result of an interacting system of bosnic field investigated by Arai in [2]. The state space is given by

$$\mathcal{H} = \mathcal{R} \otimes \mathcal{F}_{\rm b}(\mathcal{S})$$

where $\mathcal{F}_{\rm b}(\mathcal{S})$ is a boson Fock space on a Hilbert space $\mathcal{S}$ and $\mathcal{R}$ is a Hilbert pace. Let R be a self-adjoint operator on $\mathcal{R}$ with bounded from below and S be a self-adjoint and non-negative operator on $\mathcal{S}$. The free Hamiltonian is given by

$$H_0 = R \otimes I + I \otimes d\Gamma_{\rm b}(S),$$

and the total Hamiltonian is given by

$$H = H_0 + H_{\rm I},$$

where $H_{\rm I}$ is a symmetric operator on $\mathcal{H}$. Here we suppose the following conditions :

(S.1) H is self-adjoint on $\mathcal{D}(H) = \mathcal{D}(H_0) \cap \mathcal{D}(H_{\rm I})$ with bounded from below.
(S.2) For all $f \in \mathcal{D}(S) \cap \mathcal{D}(S^{-1/2})$, the weak commutators $[H_{\rm I}, I \otimes A^*(h)]^0_{\mathcal{D}(H)}$ exists. Let $\{f_n\}_{n=1}^\infty$ be the sequence of $\mathcal{D}(S) \cap \mathcal{D}(S^{-1/2})$ satisfying $\|f_n\| = 1$, $n \geq 1$. Then for all $\Psi \in \mathcal{D}(H)$ it follows that

$$\text{s-}\lim_{n\to\infty} [H_{\rm I}, I \otimes A^*(f_n)]^0_{\mathcal{D}(H)} \Psi = 0.$$

Then the next theorem follows.

Theorem A ([2] ; Theorem 1.2)
Assume **(S.1)** and **(S.2)**. Then

$$\overline{\{E_0(H) + \lambda \,\big|\, \lambda \in \sigma_{\rm ess}(S) \backslash \{0\}} \subset \sigma_{\rm ess}(H),$$

where $\overline{J}$ denotes the closure of the subset $J \subset \mathbf{R}$.

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